Quadratic equations

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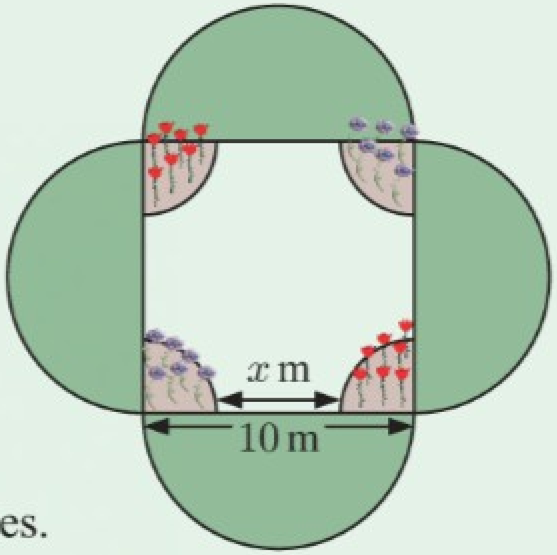
OPENING PROBLEM

A square garden gazebo is surrounded by 4 semi-circles of lawn as shown.



For a wedding, quarter circles of flowers are planted in each corner of the gazebo. The lawn covers 4 times as much area as the flower beds.

Suppose the flower beds are x metres apart as shown.

Things to think about:

- **a** Explain why the radius of each flower bed is $\left(5 - \frac{x}{2}\right)$ metres.
- **b** Hence, show that $x^2 - 20x + 50 = 0$.
- **c** By solving this equation, can you find the distance between the flower beds?

In the past we have solved many equations of the form $ax + b = 0$, $a \neq 0$. These are called **linear equations**, and have *only one* solution.

For example, $3x - 2 = 0$ is a linear equation which has the solution $x = \frac{2}{3}$.

A **quadratic equation** is an equation which can be written in the form $ax^2 + bx + c = 0$, where a, b, and c are constants, $a \neq 0$.

A quadratic equation may have *two*, *one*, or *zero* real solutions.

For example: $x^2 + 3x - 10 = 0$ is a quadratic equation.

If $x = 2$,
$$\begin{aligned} & x^2 + 3x - 10 \\ &= 2^2 + 3 \times 2 - 10 \\ &= 4 + 6 - 10 \\ &= 0 \end{aligned}$$

If $x = -5$,
$$\begin{aligned} & x^2 + 3x - 10 \\ &= (-5)^2 + 3 \times (-5) - 10 \\ &= 25 - 15 - 10 \\ &= 0 \end{aligned}$$

$x = 2$ and $x = -5$ both satisfy the equation $x^2 + 3x - 10 = 0$, so we say that they are both **solutions** or **roots** of the equation.

In contrast, the quadratic equation $x^2 + 2x + 1 = 0$ has only the one solution $x = -1$,
and the quadratic equation $x^2 + 1 = 0$ has no real solutions.

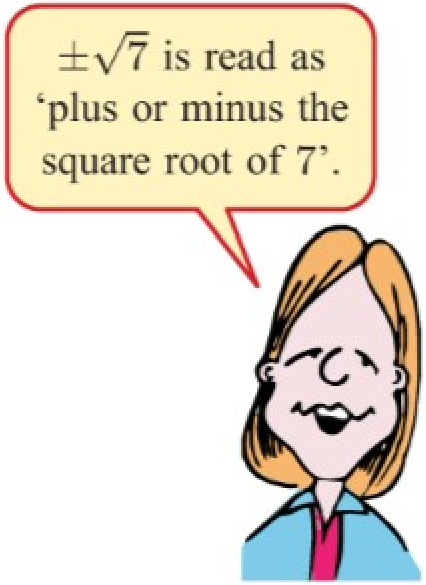
A EQUATIONS OF THE FORM $x^2 = k$

Consider the equation $x^2 = 7$.

If $x = \sqrt{7}$, then $x^2 = (\sqrt{7})^2$
$= 7$

If $x = -\sqrt{7}$, then $x^2 = (-\sqrt{7})^2$
$= (-\sqrt{7}) \times (-\sqrt{7})$
$= 7$

So, the solutions of $x^2 = 7$ are $x = \pm\sqrt{7}$.


$\pm\sqrt{7}$ is read as 'plus or minus the square root of 7'.

$$\text{If}\quad \boldsymbol{x^2 = k}\quad \text{then}\quad \begin{cases} \boldsymbol{x = \pm\sqrt{k}} & \text{if } k > 0 \\ \boldsymbol{x = 0} & \text{if } k = 0 \\ \text{there are } \textbf{no real solutions} & \text{if } k < 0. \end{cases}$$

Example 1 **Self Tutor**

Solve for x:

a $2x^2 + 1 = 15$

b $2 - 3x^2 = 8$

If $\boldsymbol{x^2 = k}$ then $\begin{cases} \boldsymbol{x = \pm\sqrt{k}} & \text{if } k > 0 \\ \boldsymbol{x = 0} & \text{if } k = 0 \\ \text{there are } \textbf{no real solutions} & \text{if } k < 0. \end{cases}$

Example 1 **Self Tutor**

Solve for x:

a $2x^2 + 1 = 15$

b $2 - 3x^2 = 8$

a $2x^2 + 1 = 15$

$\therefore\ 2x^2 = 14$ {subtracting 1 from both sides}

$\therefore\ x^2 = 7$ {dividing both sides by 2}

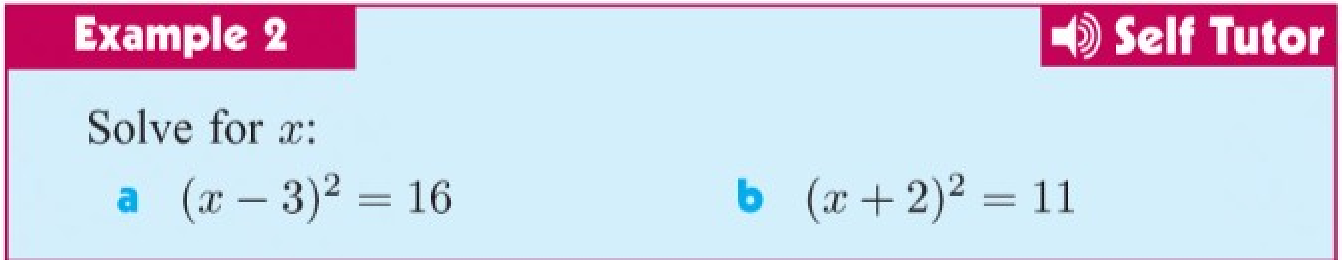
$\therefore\ x = \pm\sqrt{7}$

b $2 - 3x^2 = 8$

$\therefore\ -3x^2 = 6$ {subtracting 2 from both sides}

$\therefore\ x^2 = -2$ {dividing both sides by -3}

which has no real solutions as x^2 cannot be < 0.



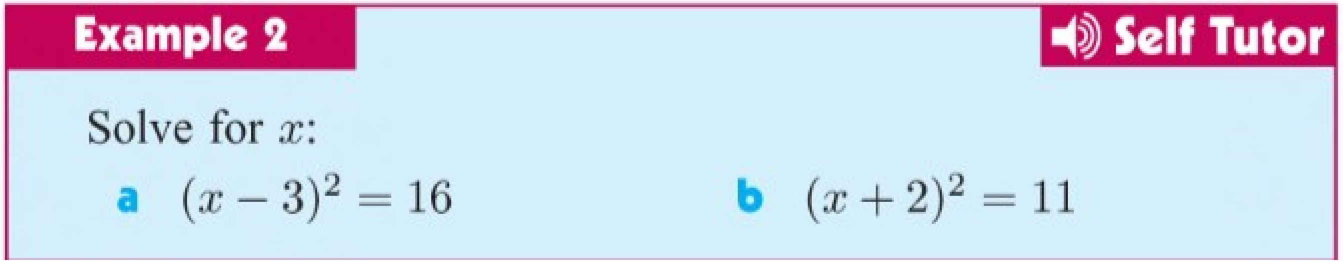
Example 2

Self Tutor

Solve for x:

a $(x-3)^2 = 16$

b $(x+2)^2 = 11$



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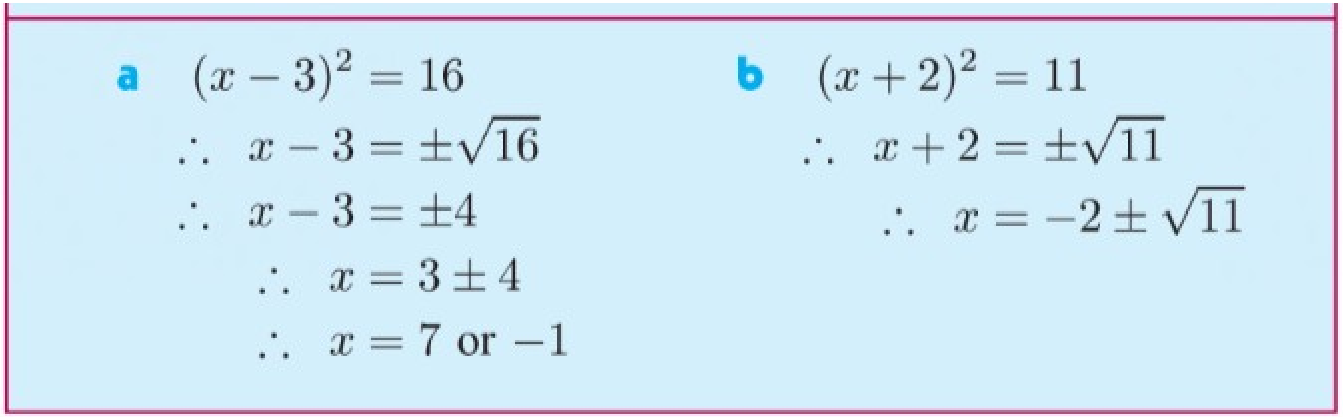

a $(x-3)^2 = 16$

$$\therefore \quad x-3 = \pm\sqrt{16}$$

$$\therefore \quad x-3 = \pm 4$$

$$\therefore \quad x = 3 \pm 4$$

$$\therefore \quad x = 7 \text{ or } -1$$

b $(x+2)^2 = 11$

$$\therefore \quad x+2 = \pm\sqrt{11}$$

$$\therefore \quad x = -2 \pm \sqrt{11}$$

1 Solve for x:

a $x^2 - 10 = 90$
b $2x^2 = 50$
c $5x^2 = 20$
d $6x^2 = 54$
e $5x^2 = -45$
f $7x^2 = 0$
g $3x^2 - 2 = 25$
h $4 - 2x^2 = 12$
i $4x^2 + 2 = 10$

2 Solve for x:

a $(x-1)^2 = 9$
b $(x+4)^2 = 16$
c $(x+2)^2 = -1$
d $(x-4)^2 = 5$
e $(x-6)^2 = -4$
f $(x+2)^2 = 0$
g $(2x-5)^2 = 0$
h $(3x+2)^2 = 4$
i $(3x+1)^2 = 81$
j $(2x+1)^2 = 48$
k $(3-2x)^2 = 7$
l $\frac{1}{3}(2x+3)^2 = 2$

3 Solve for x by first eliminating the algebraic fractions:

a $\frac{x}{4} = \frac{1}{x}$ **b** $\frac{5}{x} = \frac{x}{2}$ **c** $\frac{x}{8} = \frac{2}{x}$

4 Solve for x:

a $(x+1)^2 = (3-x)^2$ **b** $(2x-1)^2 = (2-3x)^2$

1 **a** $x = \pm 10$ **b** $x = \pm 5$ **c** $x = \pm 2$
d $x = \pm 3$ **e** no real solution **f** $x = 0$
g $x = \pm 3$ **h** no real solution **i** $x = \pm\sqrt{2}$

2 **a** $x = 4$ or -2 **b** $x = 0$ or -8 **c** no real solution
d $x = 4 \pm \sqrt{5}$ **e** no real solution **f** $x = -2$
g $x = 2\frac{1}{2}$ **h** $x = 0$ or $-\frac{4}{3}$ **i** $x = \frac{8}{3}$ or $-\frac{10}{3}$
j $x = -\frac{1}{2} \pm 2\sqrt{3}$ **k** $x = \dfrac{3 \pm \sqrt{7}}{2}$ **l** $x = \dfrac{\pm\sqrt{6} - 3}{2}$

3 **a** $x = \pm 2$ **b** $x = \pm\sqrt{10}$ **c** $x = \pm 4$

4 **a** $x = 1$ **b** $x = \frac{3}{5}$ or 1